10. Präsenzübung, **Statistische Physik**

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Aufgabe P24 *Ginzburg-Landau phenomenological model for superconductivity*

The detailed local state of a (3d) superconductor can be described by a local order parameter which is a complex-valued field $\psi(x) \in \mathbb{C}$. Although it bears some resemblance to a quantum wave function, it is not a pure quantum state. Its modulus square can be interpreted as the local density $n(x) = |\psi(x)|^2$ of the microscopic carriers of the super current, so that $\psi(x) = 0$ for all x corresponds to the normal state.
The free energy at temperature τ as function of $\psi(x)$ and $A(x)$ has the form

$$F(\psi, A) = F_0 + \alpha(\tau)\|\psi\|^2 + \frac{\beta}{2}\|\psi^2\|^2 + \frac{1}{2m}\|(-i\hbar\nabla + qA)\psi\|^2 + \frac{1}{2\mu_0}\|\mathrm{curl}A\|^2 \qquad (1)$$

where

$$\|\psi\|^2 := \int d^3x\, |\psi(x)|^2,$$

and for a real vector field $v_i(x)$,

$$\|v\|^2 := \int d^3x \sum_{i=1}^{3} v_i(x)^2.$$

These integral are performed over the finite volume V of the material. The parameter m is the effective mass and q the charge of the microscopic supercurrent carriers and

$$\alpha(\tau) = \alpha_0(\tau - \tau_c)$$

where $\alpha_0 > 0$ and $\tau_c > 0$. Here $\beta > 0$ is *not* the inverse temperature, but instead a positive phenomenological constant. F_0 is the free energy of the normal phase, which we assume independent of τ, and ∇ is the gradient operator, $A(x)$ the electromagnetic potential (a real vector field). [curl denotes "die Rotation eines Vectors".]
The equilibrium state of the field $\psi(x)$ is that which minimizes the free energy functional $F(\psi)$.

a. Characterize all the homogeneous (i.e. constant in space) equilibrium values of ψ assuming that there is no electromagnetic field ($A(x) = 0$ for all x), as a function of α_0 and τ.

b. Show that, assuming trivial boundary conditions, F is stationary with respect to a variation in ψ when

$$\alpha\psi(x) + \beta|\psi(x)|^2\psi(x) + \frac{1}{2m}(-i\hbar\nabla + qA)^2\psi = 0, \qquad (2)$$

where $(-i\hbar\nabla + qA)^2$ represents a twice consecutive application of the differential operator $(-i\hbar\nabla + qA)$.

c. Let ψ_∞ denote an equilibrium solution found in point (a). We consider small real variations from that ideal solution of the form

$$\psi(x) = \psi_\infty(1 - g(x))$$

where $g(x)$ is real. Show that, for $A = 0$, linearizing Equation (2) yields the equation

$$\nabla^2 g(x) = -\frac{4m\alpha}{\hbar^2} g(x). \tag{3}$$

d. In the superconducting phase, and under the assumption that g varies only along one direction, say $x = (0, 0, z)$, show that

$$g(0, 0, z) = g(0)e^{-\sqrt{2}z/\xi}$$

is a solution, and determine the *coherence length* ξ.

e. Assuming again that ψ is homogeneous, show that $F(\psi, A)$ is stationary with respect to local variations of the magnetic potential $A(x)$ if

$$J = -\frac{q^2}{m}|\psi|^2 A$$

where J is the current density, related to the magnetic field $B = \operatorname{curl} A$ via the static Maxwell equation $\mu_0 J = \operatorname{curl} B$.

This is one of the *London equations*, which explains why any magnetic field is automatically screened from the bulk of a superconductor by the supercurrents, in the same way that an electric field is screened from within a normal conductor by the distribution of charges.